**Brillioun Wigner Perturbation Theory**

Alternatively, we can use BW perturbation theory. This procedure is off ‘mass shell’. In BW we have,



which is similar to the RSTIPT formulas, but we’ll note that the exact energy En enters in on the right hand side, where before En(0) would have, and we’ll note the terms in this series are less complicated. The disadvantage here is that we have a much tougher equation to solve for En. An advantage is that the equation here is simpler, doesn’t require modification just ‘cause we have degeneracies, and sometimes we need only go out to finite order to get the exact result. We’ll also see this equation again when we get to time-dependent perturbation theory.

This series can be derived as follows. Let’s see…first, to start the recursion equation we need to say something like |ψn> = |ψn(0)> + O(V). We can’t do the thing we did last time where we say |ψn> = |ψn(0)> + |ψn(1)> + … and solve for each term order by order because requires us to expand En too. Instead we’ll just deliberately project out |ψn(0)> from |ψn>.



and then look for an equation for |ψn>. So to that end, we’ll go to the Schrodinger equation:



So now we have:



and we can feed the RHS into the last term to get, recursively,



and could do it again, and again, etc., to generate more terms in the expansion,



Could keep going, but I’ll stop at ‘second order’.



Now we insert a complete set of H0 (and note the Q just takes out the i, j = n term those complete sets) to get:



which is what we had above. It’s customary to set C = 1, though with the understanding that it’s not really, and we’ll still have to normalize the wavefunction in the end. Apropos the eigenvalues, we could try En = <ψn|H|ψn>, but that’s not as simple as….



and so,



which also matches. As aforementioned, the BW equations must be solved self-consistently (meaning the new En appears on both sides of the equation). This increase in complexity comes with some compensatory increase in information. For instance, the solution of the BW equation, En, is probably complex. The real part that would give us the energy shift.



But the imaginary part also has an interpretation…as the ½ the decay rate of the state. This is clearer-ish by a rudimentary argument. Say that our state ψn has the energy En. Then its evolution will be something like,



from which we recognize the oscillatory phase part as the energy. And when we take the modulus of this factor, we get:



The exponent we’d recognize as the decay rate of the state (presuming Im(En) < 0). And the lifetime of the state, τ, would be 1/decay rate. So,



Of course the state cannot decay to zero as its probability must be conserved, but basically, we’re saying the new state’s resemblance to the old state is decaying to zero. States with non-zero decay rates are called metastable states. Of course true eigenstates should have a zero decay rate, and so we expect that there are solutions, ω, to the equation, which are entirely real.

**Convergence of Perturbative Series**

We might expect that we can generate the RSTIPT eigenvalue equation from the BWTIPT equation, by saying En(V) = En(0) + En(1) + En(2) + … = En(0) + ΔEn(V) , plugging it into the BWTIPT expansion, and equating orders [implicit powers of V].



To equate orders we’d first have to Taylor series the denominators in powers of ΔEn(V). Consider the first denominator term that we’d be expanding, and suppose En(0) < Ei(0). Then for small perturbations, we can still say En(V) < Ei(0) → ΔEn(V) < Ei(0)­­ – En(0).



This Taylor series expansion won’t converge if ΔEn(V) > Ei(0) – En(0) → En(V) > Ei(0). So in other words, if a perturbation were strong enough to make one energy level cross the other (level crossing), the perturbative series won’t converge (meaning the RSTIPT series, not sure if/how the BWTIPT series is affected). Level crossing is indicative of a phase transition. This can be a problem sometimes when we have many interacting particles. As we increase N, the perturbation will grow in strength. Also, as we increase L, the unperturbed energies (of say particles in a box En ~ k2/2m, where k = 2πn/L) will get closer and closer together. For the δ(x) interaction, the finite N, L situation PT is correct to all orders. Perhaps the finite N, large L is correct too? But the TD limit is correct only up to first order. So there may be an order where PT stops being correct, or even converging – past λ = 1, or 2, etc.